

## Instability of condensation in the zero-range process with random interaction

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The zero-range process is a stochastic interacting particle system that is known to exhibit a condensation transition. We present a detailed analysis of this transition in the presence of quenched disorder in the particle interactions. Using rigorous probabilistic arguments, we show that disorder changes the critical exponent in the interaction strength below which a condensation transition may occur. The local critical densities may exhibit large fluctuations, and their distribution shows an interesting crossover from exponential to algebraic behavior.

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The zero-range process is a stochastic lattice gas where the particles hop randomly with an on-site interaction that makes the jump rate dependent only on the local particle number. It was introduced in [1] as a mathematical model for interacting diffusing particles, and since then, has been applied in a large variety of contexts, often under different names (see, e.g., [2] and references therein). The model is simple enough for the steady state to factorize; on the other hand, it exhibits an interesting condensation transition under certain conditions. *Viz.*, when the particle density exceeds a critical value  $\rho_c$ , the system phase separates into a homogeneous background with density  $\rho_c$  and all the excess mass concentrates on a single lattice site. This has been observed and studied in some detail in experiments on shaken granular media [3,4], and there is a well-established analogy to Bose-Einstein condensation [2,5,6]. It is also relevant as a generic mechanism for phase separation in single-file diffusion [7] and condensation phenomena in many complex systems such as network rewiring [8] or traffic flow [9] (for a review see [2]).

The transition can be caused by site-dependent jump rates  $g_x$  [5] due to the slowest site acting as a trap. It also appears in a more subtle fashion in homogeneous systems where condensation may result from the growth of large clusters at the expense of small clusters if the jump rates  $g(n)$  as a function of the number  $n$  of particles on the starting site have a decreasing tail. Such a model with a generic power-law decay

$$g(n) = 1 + b/n^\sigma, \quad (1)$$

with positive interaction parameters  $b$  and  $\sigma$ , has been introduced in [10]. Condensation occurs if  $0 < \sigma < 1$  and  $b > 0$  or if  $\sigma = 1$  and  $b > 2$ . The condensation transition is understood also rigorously in the context of the equivalence of ensembles [11,12], and more recently many variants of (1) have been studied [2,6,13–16].

All previous studies of zero-range processes assume that the interaction between particles is strictly equal on all sites. In this paper we study the effect of disorder on this interaction and show that even a small random perturbation of the

$n$  dependence of the jump rates  $g(n)$  leads to a drastic change in the critical behavior. Namely, for positive  $b$  condensation only occurs for  $0 < \sigma < 1/2$  (see Fig. 1), excluding in particular the frequently studied case  $\sigma = 1$ . Moreover, the critical densities are site-dependent random variables with non-trivial distributions, and depending on the parameter values, they may exhibit large fluctuations. Due to the wide applicability of the zero-range process, the change of the critical interaction exponent  $\sigma$  is particularly relevant for applications as is explained later.

We consider a lattice  $\Lambda_L$ , which we take to be periodic and of finite size  $|\Lambda_L| = L$ . A configuration is denoted by  $(\eta_x)_{x \in \Lambda}$  where  $\eta_x \in \{0, 1, \dots\}$  is the occupation number at site  $x$ . The dynamics is defined in continuous time, such that with rate  $g_x(\eta_x)$  site  $x \in \Lambda_L$  loses a particle, which moves to a randomly chosen target site  $y$  according to some probability distribution  $p(y-x)$ . For example, in one dimension with nearest-neighbor hopping, the particle moves to the right (left) with probability  $p(1-p)$ .

A generic perturbation of the jump rates (1) can be additive or multiplicative, but since the condensation behavior is determined only by the tail of the jump rates for large  $n$ , both choices are essentially equivalent. They can be written in a convenient general way,

$$g_x(n) = e^{E_x(n)} \quad \text{for } n \geq 1, \quad g(0) = 0, \quad (2)$$

where the exponents are given by

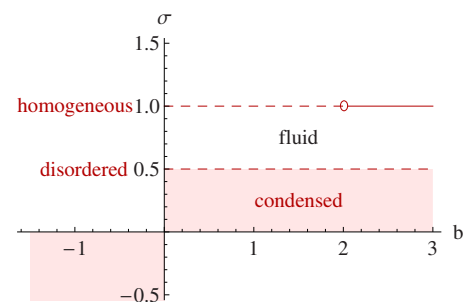


FIG. 1. (Color online) Change of the phase diagram under random perturbations of the jump rates (1). Disorder changes the critical interaction exponent from 1 to  $1/2$  and leads to a critical density that depends on the system size. Inside the red-shaded region, condensation occurs above a nonzero critical density for  $b > 0$ , and for  $b < 0$  [see (2)] and negative  $\sigma$  the critical density is zero.

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$$E_x(n) = e_x(n) + b/n^\sigma, \quad b \in \mathbb{R}, \quad \sigma > 0, \quad (3)$$

with  $e_x(n)$  being independent identically distributed random variables with respect to  $x$  and  $n$ . Without  $x$  dependence this would amount merely to a change of the function  $g$ , which might be interesting, but is a degenerate problem in terms of generic perturbations. On the other hand, the effect of spatially inhomogeneous jump rates favoring condensation on slow sites has already been studied [5]. Therefore we concentrate on disorder with spatially uniform mean  $\mathbb{E}(e_x(n)) = 0$  and variance  $\delta^2 > 0$  in order to focus on the basic novelty, which is the suppressing effect on condensation for a generic perturbation of (1). For the same reason we have chosen the jump probability  $p(y-x)$  to be homogeneous, since spatial dependence there leads to the same effect as spatially inhomogeneous jump rates [17].

Note that for  $e_x(n) \equiv 0$  (i.e.,  $\delta=0$ ) the asymptotic behavior of  $g_x(n)$  is given by (1). All data shown in this paper are for uniform  $e_x(n) \sim U(-\alpha, \alpha)$ , characterized by  $\delta^2 = \alpha^2/3$ . But our analytical results are of course independent of the distribution of the perturbation as well as the exact form of the jump rates (2). For negative  $\sigma$  the rates (2) are increasing in  $n$  for positive  $b$  and hence there is no condensation. For negative  $b$  the rates tend to zero, which means that there is condensation at critical density  $\rho_c = 0$ . This is an essentially trivial feature of the model which is robust against perturbation by disorder. We therefore focus on positive interaction exponent  $\sigma$ .

It is well known (see, e.g., [2,18]) that the process has a grand-canonical factorized steady state  $\nu_\mu^J$  with single-site marginal

$$\nu_{x,\mu}(n) = \frac{e^{n\mu}}{z_x(\mu)} \prod_{k=1}^n g_x(k)^{-1} = \frac{1}{z_x(\mu)} \exp\left(n\mu - \sum_{k=1}^n E_x(k)\right), \quad (4)$$

where the chemical potential  $\mu \in \mathbb{R}$  fixes the particle density. This holds independently of the distribution of target sites  $p(y-x)$  and for each realization of the  $e_x(k)$ ; i.e.,  $\nu_\mu^J$  is a *quenched* distribution. The single-site normalization is given by the partition function

$$z_x(\mu) = \sum_{n=0}^{\infty} \exp\left(n\mu - \sum_{k=1}^n E_x(k)\right), \quad (5)$$

which is strictly increasing and convex in  $\mu$  [11]. The local density can be calculated as usual as a derivative of the free energy,

$$\rho^x(\mu) = \langle \eta_x \rangle = \frac{\partial \ln z_x(\mu)}{\partial \mu}, \quad (6)$$

and is a strictly increasing function of  $\mu$ . By  $\langle \cdot \rangle$  we denote the (quenched) expected value with respect to  $\nu_\mu^J$  for fixed disorder; i.e., fixed realization of the  $e_x(k)$ . To study the condensation transition we have to identify the maximal or critical chemical potential  $\mu_c \in \mathbb{R}$ , such that  $z_x(\mu) < \infty$  for all  $\mu < \mu_c$ .

The single-site marginal (4) is a function of the disorder, and for  $b=0$  it has the distribution of a geometric random

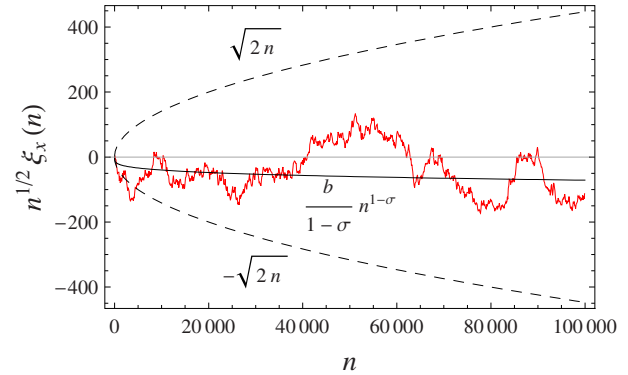


FIG. 2. (Color online) Typical realizations of  $\sqrt{n}\xi_x(n)$  (drawn with  $\delta^2=1$ ) leave the area enclosed by the dashed parabola only finitely many times, but cross the full line  $-\frac{b}{1-\sigma}n^{1-\sigma}$  for  $\sigma > 1/2$  infinitely often (here  $\sigma=0.75$ ,  $b=1$ ).

walk with deterministic drift  $\mu$ . The critical chemical potential in this case is simply  $\mu_c=0$ , and as  $n \rightarrow \infty$   $\nu_{x,\mu}(n)$  converges to a log-normal distribution. With nonzero  $b$  the drift term becomes  $n$  dependent and a more detailed analysis is required. With (3) we have, to leading order as  $n \rightarrow \infty$ ,

$$\sum_{k=1}^n E_x(k) \approx \delta \sqrt{n} \xi_x(n) + \begin{cases} \frac{b}{1-\sigma} n^{1-\sigma}, & \sigma \neq 1, \\ b \ln n, & \sigma = 1, \end{cases} \quad (7)$$

where by the central limit theorem

$$\xi_x(n) := \frac{1}{\delta \sqrt{n}} \sum_{k=1}^n e_x(k) \rightarrow N(0, 1), \quad n \rightarrow \infty, \quad (8)$$

converges to a standard Gaussian. Moreover, the process  $[\sqrt{n}\xi_x(n) : n \in \mathbb{N}]$  is a random walk with increments of mean zero and variance 1. Since the fluctuations of such a process are of order  $\sqrt{n}$ , we have, for all  $C \in \mathbb{R}$ ,

$$\mathbb{P}(\xi_x(n) \leq C \text{ for infinitely many } n) = 1 \quad (9)$$

and, for all  $\gamma > 0, C > 0$ ,

$$\mathbb{P}(|\xi_x(n)| > Cn^\gamma \text{ for infinitely many } n) = 0. \quad (10)$$

This is a direct consequence of the law of the iterated logarithm (see, e.g., [19], Corollary 14.8) and is illustrated in Fig. 2. Together with (5) and (7) this implies that  $z_x(\mu) < \infty$  for all  $\mu < 0$  with probability 1. So for almost all (in a probabilistic sense) realizations of the  $e_x(k)$  the critical chemical potential is  $\mu_c=0$  and

$$z_x(\mu_c) = \sum_{n=0}^{\infty} \exp\left[-\sum_{k=1}^n \left(e_x(k) + \frac{b}{k^\sigma}\right)\right]. \quad (11)$$

For certain values of  $\sigma$  and  $b$ ,  $z_x(\mu_c) < \infty$  is possible and  $\nu_{x,\mu_c}$  can be normalized, which is a necessary condition for a condensation transition [10]. We find the following.

$b \leq 0$ . In this case, (7) and (9) imply that there are infinitely many terms in (11) which are bounded below by 1. Since all terms of the sum are non-negative, it diverges and  $z_x(\mu_c) = \infty$  with probability 1.

$b > 0$ . In this case the asymptotic behavior of the terms in the sum (11) depends on the value of  $\sigma > 0$ , since the sign of the exponent can change.

(i) For  $\sigma > 1/2$ ,  $n^{1-\sigma} \ll \sqrt{n}$  and (7) is dominated by  $\delta\sqrt{n}\xi_x(n)$ . Applying (9) with  $C=0$  we get  $z_x(\mu_c) = \infty$  with probability 1.

(ii) For  $\sigma = 1/2$  both terms in (7) are of the same order since  $n^{1-\sigma} \sim \sqrt{n}$  and

$$-\sum_{k=1}^n \left( e_x(k) + \frac{b}{k^\sigma} \right) \approx -\delta\sqrt{n} \left( \xi_x(n) + \frac{2b}{\delta} \right). \quad (12)$$

Again, (9) this time with  $C=2b/\delta$  implies  $z_x(\mu_c) = \infty$  with probability 1.

(iii) For  $0 < \sigma < 1/2$  we have  $n^{1-\sigma} \gg \sqrt{n}$  and (7) is dominated by  $\frac{b}{1-\sigma}n^{1-\sigma}$ . We apply (10) for  $\gamma = 1 - \sigma - 1/2 > 0$  to see that the random quantity  $\xi_x(n)$  can change the sign of the exponent in (11) only for finitely many terms in the sum. Therefore  $z_x(\mu_c) < \infty$  with probability 1 since  $\frac{b}{1-\sigma}n^{1-\sigma}$  has a fixed negative sign in (11).

Whenever  $z_x(\mu_c) = \infty$  the local critical density

$$\rho_c^x := \rho^x(\mu_c) = \infty \quad (13)$$

also diverges and there is no condensation transition (see, e.g., [21], Lemma I.3.3). But for  $b > 0$  and  $0 < \sigma < 1/2$  we have  $z_x(\mu_c) < \infty$ , and by the same argument as above it follows that

$$\rho_c^x = \frac{1}{z_x(\mu_c)} \sum_{n=0}^{\infty} n \exp \left[ -\sum_{k=1}^n \left( e_x(k) + \frac{b}{k^\sigma} \right) \right] < \infty, \quad (14)$$

with probability 1, since the factor  $n$  in the sum only gives a logarithmic correction in the exponent. Therefore there is condensation since the grand-canonical product measures only exist up to a total density of

$$\rho_c(L) := \frac{1}{L} \sum_{x \in \Lambda_L} \rho_c^x, \quad (15)$$

which depends on the  $e_x(n)$  and the size of the lattice,  $L$ .

If the actual number of particles,  $N$ , is larger than  $L\rho_c(L)$ , all sites except the ‘‘slowest’’ one contain on average  $\langle \eta_x \rangle_N = \rho_c^x$  particles. They form the so-called *critical background*, since their distribution has nonexponential tails. By  $\langle \cdots \rangle_N$  we denote the (canonical) expectation conditioned on the total particle number  $N$ . The slowest site, say,  $y$ , is defined by  $\rho_c^y > \rho_c^x$  for all  $x \neq y$ . By the conservation law it is required that

$$\langle \eta_y \rangle_N = N - \sum_{x \neq y} \rho_c^x = O(N); \quad (16)$$

i.e., it contains of order  $N$  particles and forms the *condensate*. This interpretation is in accordance with previous results and has been proved rigorously in [20] in the limit  $N \rightarrow \infty$  for the unperturbed model. For the perturbed model we support this conclusion by Monte Carlo (MC) simulations, some of which are shown in Fig. 3. For fixed  $L$  we plot the stationary background density

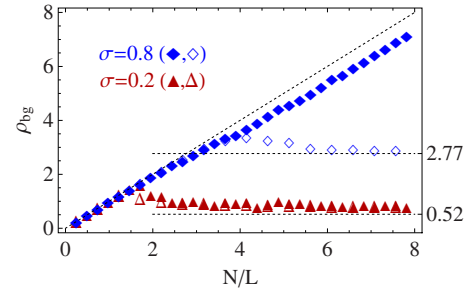


FIG. 3. (Color online) Background density  $\rho_{bg}$  Eq. (17), as a function of  $N/L$  for  $b=1.2$ . Data for fixed disorder with  $\delta^2=1/12$  and  $L=1024$  show condensation for  $\sigma=0.2$  and no condensation for  $\sigma=0.8$ . Data for  $\delta^2=0$  are shown by open symbols; dotted lines indicate the critical densities. Data points are averages over 100 MC samples with errors of the size of the symbols.

$$\rho_{bg} := \frac{1}{L} (N - \langle \eta_y \rangle_N) \quad (17)$$

as a function of the total density  $\rho = N/L$ . For  $\sigma=0.2$ ,  $\rho_{bg}$  converges to a critical density  $\rho_c(L)$  which is slightly higher than for the unperturbed model. The overshoot of  $\rho_{bg}$  for densities close to  $\rho_c(L)$  is due to sampling from the canonical rather than the grand-canonical ensemble, which has been observed already in [22]. For  $\sigma=0.8$ ,  $\rho_{bg}$  increases approximately linearly with  $\rho$ , which is clearly different from the unperturbed model condensing with critical density 2.77.

For the perturbed system the critical density  $\rho_c(L)$  is a random variable, which according to (15) converges in the thermodynamic limit to the expected value  $\mathbb{E}(\rho_c^x)$  with respect to the disorder. This is in general hard to calculate [23], even for simple choices of  $e_x(n)$ . Detailed numerical estimates of the distribution of  $\rho_c^x$  (some of which are shown in Fig. 4) indicate that depending on the system parameters the cumulative tail is either algebraic or consists of an algebraic part with an exponential cutoff at large values. This can be explained heuristically by the interplay of the two terms in (7) that determine the main contributions to the partition

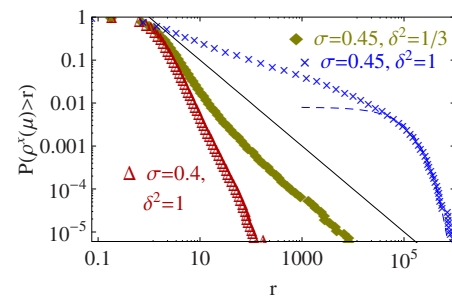


FIG. 4. (Color online) Cumulative tail of the distribution of  $\rho_c^x$  from  $5 \times 10^5$  independent numerical calculations of (14) with  $b=1$ , where infinite sums have been cut off at  $n=10^6$ . The solid line corresponds to a cumulative exponent  $-1$ . The data exhibit power-law decays and for some parameters also large exponential cutoffs: e.g.  $\sim e^{-r/10^5}$  for  $\sigma=0.45$ ,  $\delta^2=1$  fitted by a dashed line. The expectations are finite but may be very large,  $\mathbb{E}(\rho_c^x) = 1.60$  ( $\Delta$ ), 2.55 ( $\diamond$ ), 1106 ( $\times$ ).

function [24]. The exponents of the purely algebraic tails are smaller than but often close to  $-1$ , and the length scale of the exponential tails can be extremely large (cf. Fig. 4). Therefore  $\rho_c^x$  has a finite mean which determines the thermodynamic limit of (15),

$$\rho_c := \lim_{L \rightarrow \infty} \rho_c(L) = \mathbb{E}(\rho_c^x) < \infty, \quad (18)$$

but for finite  $L$  the  $\rho_c^x$  can exhibit large fluctuations, resulting in high values for  $\rho_c$  and slow convergence of (18).

The interaction encoded in the jump rates of a zero-range model represents an effective interaction for which space dependence and randomness due to microscopic impurities or heterogeneities in the case of complex systems has to be

taken into account. We have shown that generic interaction disorder reduces the critical interaction exponent from  $\sigma=1$  to  $\sigma=1/2$ . In particular, this implies that for the most-studied case  $\sigma=1$  there is no condensation transition in the presence of interaction disorder. This case is relevant for the mapping of the zero-range process to exclusion models where it becomes an effective model for domain wall dynamics and therefore a powerful criterion for phase separation in more general particle systems [7]. In this mapping the spatial interaction disorder maps onto hopping rates which depend in a random fashion on the interparticle distance. Then our results imply that in such heterogeneous finite systems the change of the critical interaction exponent has to be taken into account and finite-size effects may play a major role due to large fluctuations of local critical densities.

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